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On a Problem of Rivlin in L_p $(1 < p < \infty)$ Approximation*

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A partial answer to a problem of Rivlin ("Abstract Spaces and Approximation," Berkhäuser Verlag, Basel/Stuttgart, 1969) in L_p $(1 < p < \infty)$ approximation is given.

1. INTRODUCTION

In a conference held at Oberwolfach in 1968, Rivlin [1] proposed the following problem:

Characterize those n-tuples of algebraic polynomials $\{p_0, p_1,..., p_{n-1}\}$ with degrees satisfying

$$\deg p_j = j \qquad (j = 0, 1,..., n-1)$$

for which there exists an $f \in C[-1, 1]$ such that the polynomial of best uniform approximation of degree j to f is p_j $(j = 0, 1,..., n-1)$.

Several authors have studied this problem (see the references in [2]). In this paper we consider the above problem in $C[-1, 1]$ with the L_p $(1 < p < \infty)$ norms and give a partial answer.

THEOREM 1. *Let V_1 and V_2 be Chebyshev subspaces with dimensions m and n $(m < n)$, respectively. Let $V_1 \subset V_2$ and $v_j \in V_j$ $(j = 1, 2)$. If there exists an $f \in C[-1, 1]$ such that v_j is a best L_p $(1 < p < \infty)$ approximation to f from V_j $(j = 1, 2)$, then the function $v = v_2 - v_1$ changes sign at least m times in $[-1, 1]$.*

However, in general, this condition is not sufficient, because we have

THEOREM 2. *Let V_1 and V_2 be subspaces of $C[-1, 1]$. Let $V_1 \subset V_2$ and*

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$v_j \in V_j$ $(j=1,2)$. Then there exists an $f \in C[-1,1]$ such that v_j is a best L_2 approximation to f from V_j $(j=1,2)$ if and only if v_1 is a best L_2 approximation to v_2 from V_1.

Some notation is needed in the following proof:

$$Z_+(g)=\{x\in[-1,1]: g(x)>0\},$$
$$Z_-(g)=\{x\in[-1,1]: g(x)<0\},$$
$$Z(g)=\{x\in[-1,1]: g(x)=0\}.$$

2. Proof of Theorem 1

Suppose on the contrary that there exists an $f\in C[-1,1]$ such that v_j is a best approximation to f from V_j $(j=1,2)$, but the condition of Theorem 1 is not satisfied, i.e., v changes sign at most $m-1$ times. In this case we claim that for every $x\in[-1,1]$ we have either

$$f(x)\leqslant \min\{v_1(x), v_2(x)\} \qquad \text{or} \qquad f(x)\geqslant \max\{v_1(x), v_2(x)\}. \tag{1}$$

Whence, assume that $Z(v)\cap(-1,1)=\{x_1,\ldots,x_k\}$ and $x_0=-1<x_1<\cdots<x_k<x_{k+1}=1$, then $f-v_2$ does not change sign on each of the intervals $[x_i,x_{i+1}]$, $i=0,1,\ldots,k$. Thus $f-v_2$ changes sign at most k times. Since $k<n$, this is a contradiction [3, Theorem 4].

The remainder of the proof is devoted to showing (1). It is well known that

$$\int_{[-1,1]} u\,|f-v_j|^{p-1}\operatorname{sgn}(f-v_j)\,dx=0, \qquad \forall u\in V_j,\quad j=1,2$$

or

$$\int_{Z_+(f-v_j)} u\,|f-v_j|^{p-1}\,dx=\int_{Z_-(f-v_j)} u\,|f-v_j|^{p-1}\,dx, \qquad \forall u\in V_j,\quad j=1,2. \tag{2}$$

Since v changes sign at most $m-1$ times in $[-1,1]$, there exists a $w\in V_1$ such that

$$\operatorname{sgn} w(x)=\operatorname{sgn} v(x), \qquad \forall x\in[-1,1]\setminus Z(v).$$

We first examine several simple inequalities

$$\int_{Z_+(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_1|^{p-1}\,dx \geqslant \int_{Z_+(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx, \tag{3}$$

$$\int_{Z_+(f-v_1)\cap Z(f-v_2)} w\,|f-v_1|^{p-1}\,dx \geqslant 0, \tag{4}$$

$$\int_{Z(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx \leqslant 0. \tag{5}$$

In fact, on $Z_+(f-v_1)\cap Z_+(f-v_2)$ since for $x\in Z_+(v)(Z_-(v))$ we have that $w(x)>0$ (<0) and $f(x)>v_2(x)>v_1(x)$ $(f(x)>v_1(x)>v_2(x))$,

$$w(x)\,|f(x)-v_1(x)|^{p-1} \geqslant w(x)\,|f(x)-v_2(x)|^{p-1}. \tag{6}$$

On the other hand, (6) is also valid for $x\in Z(v)$. Thus (3) holds. Inequality (4) results from that on $Z_+(f-v_1)\cap Z(f-v_2)$ $v(x)=v_2(x)-v_1(x)=f(x)-v_1(x)>0$ and, hence, $w(x)>0$. Inequality (5) results from that on $Z(f-v_1)\cap Z_+(f-v_2)$ $v(x)=v_2(x)-v_1(x)=v_2(x)-f(x)<0$ and, hence, $w(x)<0$.

Now, denoting that

$$I_j^+ = \int_{Z_+(f-v_1)\cap Z_-(f-v_2)} w\,|f-v_j|^{p-1}\,dx,$$

$$I_j^- = -\int_{Z_-(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_j|^{p-1}\,dx, \qquad j=1,2,$$

from (3)–(5) it follows that

$$\begin{aligned}
&\int_{Z_+(f-v_1)} w\,|f-v_1|^{p-1}\,dx \\
&\quad = \int_{Z_+(f-v_1)\cap Z_-(f-v_2)} w\,|f-v_1|^{p-1}\,dx + \int_{Z_+(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_1|^{p-1}\,dx \\
&\qquad + \int_{Z_+(f-v_1)\cap Z(f-v_2)} w\,|f-v_1|^{p-1}\,dx \\
&\quad \geqslant I_1^+ + \int_{Z_+(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx \\
&\quad = I_1^+ + \int_{Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx - \int_{Z(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx \\
&\qquad - \int_{Z_-(f-v_1)\cap Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx \\
&\quad \geqslant I_1^+ + I_2^- + \int_{Z_+(f-v_2)} w\,|f-v_2|^{p-1}\,dx.
\end{aligned}$$

Similarly we can obtain

$$\int_{Z_-(f-v_2)} w\,|f-v_2|^{p-1}\,dx \geqslant I_1^- + I_2^+ + \int_{Z_-(f-v_1)} w\,|f-v_1|^{p-1}\,dx.$$

By (2) the previous two inequalities imply that

$$I_1^+ + I_2^+ + I_1^- + I_2^- \leqslant 0.$$

But it is easy to see that $I_j^+ \geqslant 0$ and $I_j^- \geqslant 0$, $j=1,2$. So $I_j^+ = I_j^- = 0$, $j=1,2$. Therefore

$$Z_+(f-v_1) \cap Z_-(f-v_2) = Z_-(f-v_1) \cap Z_+(f-v_2) = \varnothing$$

and (1) is proven.

3. Proof of Theorem 2

Sufficiency is trivial, say, $f=v_2$. For necessity it is well known that

$$\int_{[-1,1]} u(f-v_j)\,dx = 0, \qquad \forall u \in V_j, \qquad j=1,2.$$

Hence

$$\int_{[-1,1]} u(v_2-v_1)\,dx = 0, \qquad \forall u \in V_1,$$

which means that v_1 is a best L_2 approximation to v_2 from V_1.

References

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